Dual solutions of the Greenspan–Carrier equations II

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Numerical integration of the boundary-layer equations associated with flow past a semi-infinite flat plate in the presence of an aligned magnetic field has shown that the solutions are not unique if $\epsilon < 1$ for certain values of $\beta < 1$, where ϵ is an intrinsic property of the fluid and β a property of conditions at infinity. An analytic explanation of this phenomenon is given here. The main properties as $\beta \to 1$ of the unique solutions when $\epsilon > 1$ are elucidated. Further, the equations associated with flow past a solid boundary in which the magnetic field is zero are solved numerically. The solutions appear to be unique but, on the other hand, the maximum value β_0, of β, for which they can be found, tends to zero with ϵ.

1. Introduction

In their study of the flow of a highly conducting, almost inviscid, incompressible fluid past a thin flat plate in the presence of a magnetic field which is uniform at infinity and parallel to the stream, Carrier & Greenspan (1959) showed that the velocity and magnetic field in the boundary layer on either side of the plate depends on the solution of the pair of linked ordinary differential equations

$$f_1''' + f_1 f_1'' - \beta g_1 g_1'' = 0, \tag{1.1a}$$

$$g_1'' + \epsilon(f_1 g_1' - g_1 f_1') = 0, \tag{1.1b}$$

subject to the boundary conditions

$$f_1(0) = f_1'(0) = g_1(0) = 0, \quad f_1'(\infty) = g_1'(\infty) = 2. \tag{1.2}$$

In these equations the independent variable

$$\eta = \tfrac{1}{2} y (U/\nu x)^{\frac{1}{2}},$$

where y measures distance from the plate, x distance along the plate from the leading edge, U is the undisturbed velocity and ν the kinematic viscosity. Further, if u, H_x are the x-components of the velocity and magnetic fields,

$$u = \tfrac{1}{2} U f_1'(\eta), \quad H_x = \tfrac{1}{2} H_0 g_1'(\eta),$$

where H_0 is the strength of the undisturbed magnetic field. Finally

$$\beta = H_0^2/4\pi\rho U^2, \quad \epsilon = 4\pi\sigma\nu, \tag{1.3}$$

where ρ is the density, σ the conductivity of the fluid, whose permeability is taken to be unity and where Gaussian units have been used throughout. The boundary conditions in (1.2) are familiar to students of boundary-layer theory except possibly $g_1(0) = 0$ which is necessary by symmetry and the assumption that the plate is of zero thickness.

The solution of (1.1) subject to (1.2) we shall refer to as Problem I. It has been shown (Carrier & Greenspan 1959; Reuter & Stewartson 1961; Meksyn 1962) that it has no solutions for all $\epsilon > 0$ if $\beta > 1$, while Glauert (1961) has examined the solution when $\epsilon \gg 1$ and $\epsilon \ll 1$. In a very recent paper with the same title as this one, Wilson (1964) has shown numerically that if $0 < \epsilon < 1$ the solution of Problem I is not unique for a range of values of β and in this paper we wish to offer some analytic arguments in support of his work.

A second problem, Problem II, can be set up by changing the boundary conditions in (1.2) to

$$f_2(0) = f_2'(0) = g_2'(0) = 0, \quad f_2'(\infty) = g_2'(\infty) = 2, \tag{1.4}$$

where the functions f_2, g_2 otherwise have identical properties to f_1, g_1. This problem arises in considering the flow past a thin non-conducting cylinder (but not of zero thickness) on the assumption that the magnetic field inside the cylinder is zero. This model, proposed by Sears & Resler (1959) has been the subject of some controversy (e.g. Stewartson 1963), but since it has not yet been completely resolved there is some value attached to studying Problem II in this context.

It proves convenient to study

$$f''' + ff'' - gg'' = 0, \tag{1.5a}$$

$$g'' + \epsilon(fg' - gf') = 0, \tag{1.5b}$$

subject to

$$f(0) = f'(0) = 0, \quad f''(0) = p, \tag{1.6}$$

and either

$$g(0) = 0, \quad g'(0) = q \tag{1.7a}$$

or

$$g(0) = r, \quad g'(0) = 0, \tag{1.7b}$$

from which the solutions of Problems I or II can be found by suitable scaling operations (Wilson 1964). This was, in fact, how the numerical integrations were carried out, and it is noted that without loss of generality q or r may be set equal to unity.

The division of work in this paper is that one of us (D. H. W.) is responsible for the numerical integrations and the other (K. S.) for the analysis.

2. Classification of singularities

Since the numerical procedure adopted here and explained in the earlier paper by Wilson (1964) is step-by-step, it is important to know the singularity, either at $\eta = \infty$ or a finite value, which terminates the process. The classification which we now give seems to be complete, except possibly when $\epsilon = 1$, but it must be borne in mind that our approach is heuristic and we cannot prove the list to be all-inclusive.

(a) *The singularity in f is independent of g*

Here we assume that near the singularity gg'' may be neglected in (1.5a) so that f satisfies

$$f''' + ff'' = 0. \tag{2.1}$$

An appropriate form for f near the singularity is

$$f = A_1(\eta - \eta^*)^{\alpha_1}, \tag{2.2}$$

where α_n, A_n, η^* are constants: if we are considering the behaviour of f near a singularity at a finite value of η, that value is η^*, but (2.2) is also relevant if we are considering the behaviour of f as $\eta \to \infty$. On substituting in (2.1) we find that either

$$\alpha = 0, \quad A_1 \quad \text{arbitrary}, \tag{2.3a}$$

or

$$\alpha = 1, \quad A_1, \quad \eta^* \quad \text{arbitrary}, \tag{2.3b}$$

or

$$\alpha = -1, \quad A_1 = 3, \quad \eta^* \quad \text{arbitrary}. \tag{2.3c}$$

In the first two cases the singularity clearly occurs at infinity. On checking for consistency by substituting (2.2), (2.3a) or (2.3b) into (1.5b) we find that it is obtained in general of $A_1 > 0$. For $A_1 < 0$ consistency is only possible if $g'' \equiv 0$.

In the third case the behaviour of g near $\eta = \eta^*$ is found by substituting (2.2) into (1.5b) which becomes

$$(\eta-\eta^*)^2 g'' + 3\epsilon[(\eta-\eta^*)\,g' + g] = 0 \tag{2.4}$$

near $\eta = \eta^*$; therefore

$$g \sim B_1(\eta-\eta^*)^{\gamma}, \tag{2.5}$$

where B_n is arbitrary and

$$\gamma(\gamma-1) + 3\epsilon(\gamma+1) = 0,$$

i.e. $\gamma = \gamma_1, \gamma_2$ where

$$\gamma_1 = -\tfrac{1}{2}(3\epsilon-1) + \tfrac{1}{2}[9(\epsilon-1)^2 - 8]^{\frac{1}{2}}, \quad \gamma_2 = -\tfrac{1}{2}(3\epsilon-1) - \tfrac{1}{2}[9(\epsilon-1)^2-8]^{\frac{1}{2}}. \tag{2.6}$$

If $0 \leqslant \epsilon \leqslant 1 - \frac{2}{3}\sqrt{2}$, both values of γ are real and greater than -1; hence $|gg''| \ll (\eta-\eta^*)^{-4}$ as $\eta \to \eta^*$ and may be neglected, in the neighbourhood of $\eta = \eta^*$, in (1.5a). If $1 - \frac{2}{3}\sqrt{2} < \epsilon < 1$, γ is complex so that g oscillates either finitely or infinitely near $\eta = \eta^*$ but $\operatorname{Re}\gamma > -1$, implying that gg'' is still negligible in (1.5a). The assumptions about the singularity are therefore consistent for all $\epsilon < 1$. Further, it is known for (1.7a) and by extension for (1.7b) (Reuter & Stewartson 1961) that if $p > 0, f'' > 0$ for all η. Hence the singularity characterized by (2.3c) cannot occur if $p > 0$.

On the other hand if $\epsilon > 1$ at least one of the possible values of γ has a real part less than -1 which implies that as $\eta \to \eta^*$, $|gg''| \gg |ff''|$ in general, which is a contradiction.

It is concluded that the singularity described by (2.3c) (2.6) is only possible if $\epsilon < 1$ and $p < 0$.

(*b*) *The singularities in f, g are interdependent*

This is only possible in general if $f \approx g$ near the singularity. We assume, therefore, that

$$f \simeq A_2(\eta-\eta^*)^{\alpha_2}, \quad f - g \simeq C_2(\eta-\eta^*)^{\delta_2}, \tag{2.7}$$

near the singularity where C_n, δ_n are constants, $\alpha_2 > \delta_2$ if $\alpha_2 > 0$ and $\alpha_2 < \delta_2$ if $\alpha_2 < 0$. If the singularity occurs at a finite value of η, that value is to be η^*. On substituting into (1.5b) we have

$$A_2\alpha_2(\alpha_2-1)\,(\eta-\eta^*)^{\alpha_2-2} = \epsilon A_2 C_2(\alpha_2-\delta_2)\,(\eta-\eta^*)^{\alpha_2+\delta_2-1}$$

so that

$$\delta_2 = -1 \quad \text{and} \quad C_2 = -\alpha_2(\alpha_2-1)/\epsilon(\alpha_2+1). \tag{2.8}$$

Further, on substituting into (1.5a), (2.7) is consistent provided

$$\alpha_2(\alpha_2-1)(\alpha_2-2) = \alpha_2(\alpha_2-1)[\alpha_2(\alpha_2-1)+2]/\epsilon(\alpha_2+1),$$

i.e.
$$\alpha_2 = 0, \tag{2.9a}$$
or
$$\alpha_2 = 1, \tag{2.9b}$$
or
$$\alpha_2 = \alpha_3, \alpha_4, \tag{2.9c}$$
where
$$\alpha_3 = \tfrac{1}{2}+\tfrac{1}{2}[(9\epsilon+7)/(\epsilon-1)]^{\frac{1}{2}}, \quad \alpha_4 = \tfrac{1}{2}-\tfrac{1}{2}[(9\epsilon+7)/(\epsilon-1)]^{\frac{1}{2}}. \tag{2.10}$$

The first two cases require that the singularity occurs at infinity and are otherwise identical with (2.3a), (2.3b). Thus $A_2 > 0$.

If $\epsilon > 1$ the positive root α_3 indicates a singularity at infinity, the value of η^* being irrelevant to its character. On the other hand, the negative root α_4 indicates a singularity at a finite value of η and it is noted that $|\alpha_3|, |\alpha_4| > 1$ so that we have consistency. In both of these cases $f'' > 0$ near $\eta = \eta^*$ is possible, since A_2 is arbitrary, and they can therefore arise if $p > 0$ in (1.6).

If $\epsilon < 1$ both roots for α are complex so that f', g'' oscillate as $\eta \to \eta^*$. Hence they cannot arise in a numerical integration with $p > 0$. Further, since $\operatorname{Re}\alpha = \frac{1}{2}$, $f \approx g$ near the singularity only if it occurs at infinity.

3. Dual solutions of Problem I. (i) $0 < \epsilon < 1$

Although the numerical solutions obtained were carefully checked the existence of dual solutions of (1.1), (1.2) is certainly surprising and in order to promote confidence in their existence we offer in this and the next section some analytic arguments in support of the computations. The basis for the arguments is that the classification of singularities in the previous section is complete, which means that if $f''(0) > 0, f'$ and g' must tend to limits as $\eta \to \infty$.

Suppose we fix $q = 1$ in (1.7a) and let p decrease from infinity. So long as p is not small we can expect $g'(\infty) \sim 1$ while $f'(\infty) \sim p^{\frac{2}{3}}$ when p is large. Consequently, the appropriate value in (1.1a) of

$$\beta \equiv [g'(\infty)/f'(\infty)]^2 \tag{3.1}$$

$\sim p^{-\frac{4}{3}}$ as $p \to \infty$; we can expect it to increase initially as p decreases from infinity and to tend to a limit $\beta_0(\epsilon)$ as $p \to 0$. In this section we shall discuss the limit solution as $p \to 0$ and in the next we shall show that $\max\beta > \beta_0$ when $\epsilon \ll 1$.

Here, then, $p \ll 1$. Write

$$p = 2\epsilon\eta_0^3\exp(-\tfrac{1}{2}\eta_0^2\sqrt{\epsilon}), \tag{3.2}$$

where $\eta_0 \gg 1$. If $\eta = O(1)$ the variation of g' can be neglected, since $g'' = O(p)$, and so can ff''. The governing equations then reduce to

$$f''' = \eta g'', \quad g''' = \epsilon\eta f'' \tag{3.3}$$

with solution

$$f''' = \epsilon\eta_0^3\exp(-\tfrac{1}{2}\eta_0^2\sqrt{\epsilon})[\exp(\tfrac{1}{2}\eta^2\sqrt{\epsilon})+\exp(-\tfrac{1}{2}\eta^2\sqrt{\epsilon})], \tag{3.4a}$$

$$g'' = \epsilon^{\frac{3}{2}}\eta_0^3\exp(-\tfrac{1}{2}\eta_0^2\sqrt{\epsilon})[\exp(\tfrac{1}{2}\eta^2\sqrt{\epsilon})-\exp(-\tfrac{1}{2}\eta^2\sqrt{\epsilon})]. \tag{3.4b}$$

When η is large but $O(1)$,

$$f = (\eta_0^3/\eta^2)\exp\{\tfrac{1}{2}\sqrt{\epsilon}(\eta^2-\eta_0^2)\}[1+O(\eta^{-2})], \tag{3.5a}$$

$$g = \eta+(\eta_0^3\epsilon^{\frac{1}{2}}/\eta^2)\exp\{\tfrac{1}{2}\sqrt{\epsilon}(\eta^2-\eta_0^2)\}[1+O(\eta^{-2})]. \tag{3.5b}$$

Now write
$$\eta = \eta_0 + \xi/\eta_0 \tag{3.6}$$
and suppose that $\xi = O(1)$. Then when ξ is large and negative,
$$f = \eta_0 e^{\xi\sqrt{\epsilon}}[1+O(\eta_0^{-2})], \quad g = \eta_0[1+\epsilon^{\frac{1}{2}} e^{\xi\sqrt{\epsilon}}\{1+O(\eta_0^{-2})\}] \tag{3.7}$$
from (3.5), while when $\xi \approx 1$ we must expect the terms neglected in (3.3) to be significant. Since, from (3.7) $f, g = O(\eta_0)$ when $\xi = O(1)$, write
$$f = \eta_0 F_0(\xi), \quad g = \eta_0 G_0(\xi), \tag{3.8}$$
whereupon F_0, G_0 satisfy equations identical with (1.5) subject to boundary conditions
$$F_0(-\infty) = 0, \quad G_0(-\infty) = 1 \tag{3.9}$$
in the limit $\eta_0 \to 0$.

For ξ large and negative
$$F_0 = e^{\xi\sqrt{\epsilon}} - \frac{1-\epsilon}{6\epsilon^{\frac{1}{2}}} e^{2\xi\sqrt{\epsilon}} - \frac{(1-\epsilon)(3\epsilon-5)}{144\epsilon} e^{3\xi\sqrt{\epsilon}} + \ldots,$$
$$G_0 = 1+\epsilon^{\frac{1}{2}} e^{\xi\sqrt{\epsilon}} - \frac{1-\epsilon}{12} e^{2\xi\sqrt{\epsilon}} - \frac{(1-\epsilon)(7\epsilon-5)}{432\epsilon^{\frac{1}{2}}} e^{3\xi\sqrt{\epsilon}} + \ldots. \tag{3.10}$$

These equations have been integrated numerically for $\epsilon = 0{\cdot}01$, $0{\cdot}1$, $0{\cdot}5$ and it is found, as forecast in §2, that F_0', G_0' tend to finite limits as $\xi \to \infty$. The method of integration was to use the series expansion (3.10) to obtain the values of F_0, G_0, and their derivatives at a finite value of ξ and then to continue the integration using the standard step-by-step procedure. One might expect F_0', G_0' to tend to finite limits, for initially $F_0'' > G_0''$ so that F_0 is initially increasing faster than G_0 and may therefore eventually be expected to overtake it, whereupon F_0''' will be negative and reduce F_0'' to zero. The situation is quite different if $\epsilon > 1$, for initially G_0, F_0 are drawing farther apart. It is likely that for all $\epsilon > 1$, F_0, G_0 develop singularities of the type (2.10) at some finite value of η.

The general shape of the graphs of f, g found in the numerical solution with $p \ll 1$ agrees with the above argument in having a long section in which f'', g'' are small followed by a short zone in which f'', g'' are highly peaked (Wilson 1964, figure 3) and in which the majority of the increase of f', g' occurs. In order to satisfy the boundary conditions of Problem I when $\eta_0 \gg 1$, a simple affine transformation shows that if
$$g_1'(0) = 2\eta_0^{-2}[G_0'(\infty)]^{-1}, \quad f_1''(0) = 2^{\frac{5}{2}}\epsilon \exp(-\tfrac{1}{2}\eta_0^2\sqrt{\epsilon})[F'(\infty)]^{-\frac{3}{2}}, \tag{3.11}$$
then in the limit $\eta_0 \to 0$, f_1, g_1 are solutions of (1.1) with
$$\beta = \beta_0 \equiv [G_0'(\infty)/F_0'(\infty)]^2. \tag{3.12}$$
A graph of β_0 against ϵ is shown in figure 1 for $0 < \epsilon < 1$. Further, if $\beta - \beta_0$ is small the appropriate value of η_0 may be estimated as follows. If η_0 is large, we see from (3.7) that the required form for f, g differs from (3.9) by proportionate terms $O(\eta_0^{-2})$. Consequently we expect the proportionate changes in $F_0'(\infty)$, $G_0'(\infty)$ to be $O(\eta_0^{-2})$, which implies that
$$\beta - \beta_0 = A_3\eta_0^{-2}, \tag{3.13}$$

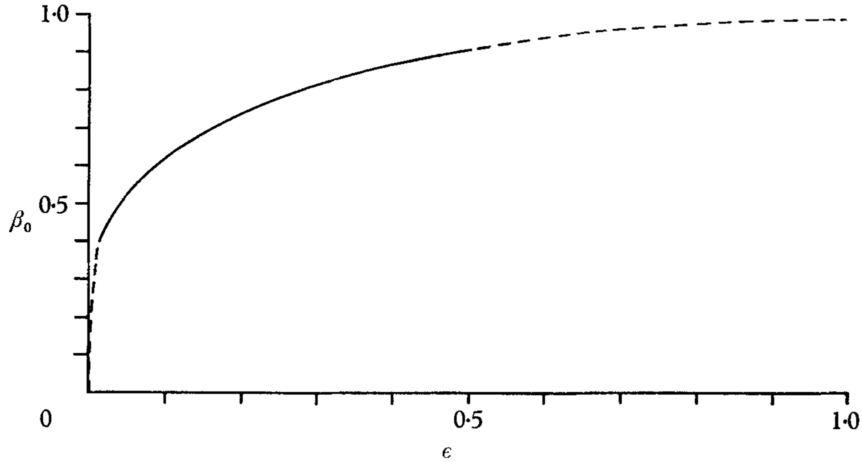

FIGURE 1. Graph of β_0, the cut-off value of β, against ϵ ($\leqslant$ 1).

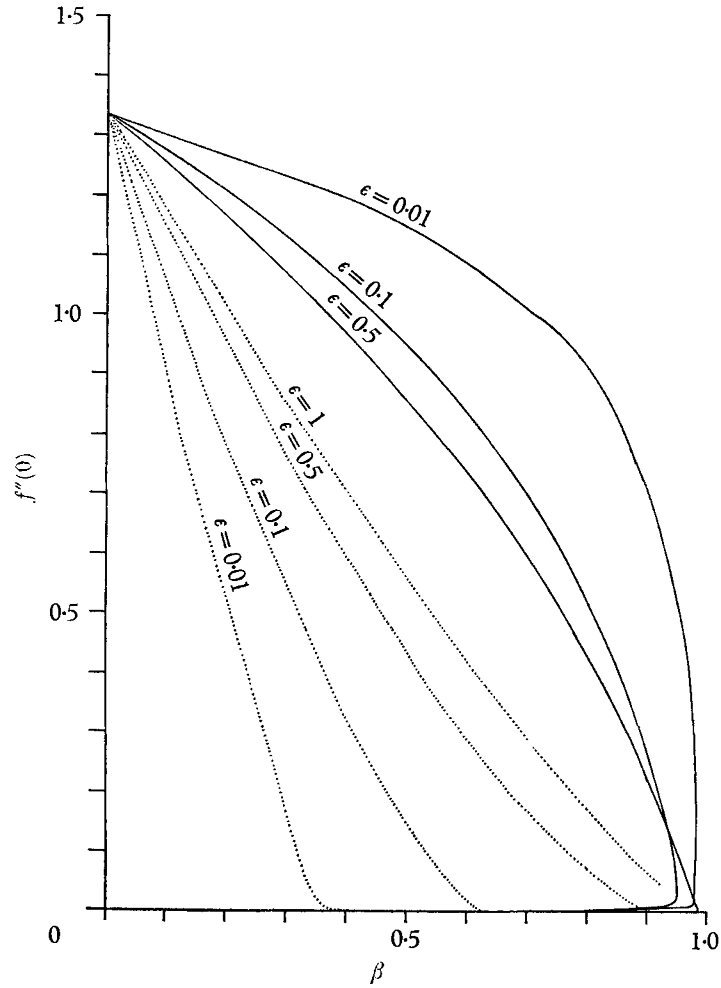

FIGURE 2. Graphs of $f''(0)$ against β for various ϵ. ——, Problem I, $g(0) = 0$; ······, Problem II, $g'(0) = 0$.

where A_3 is a function of ϵ only. Hence in Problem I, when $\beta-\beta_0$ is small,

$$f_1''(0) \propto \exp\left[-A_3\sqrt{\epsilon}/2(\beta-\beta_0)\right], \quad g_1'(0) \propto (\beta-\beta_0); \tag{3.14}$$

the factors of proportionality depend on ϵ which is assumed to be neither zero nor unity. The lower halves of the curves of $f_1''(0)$ against β displayed in figure 2 agree with (3.14) in that they have very flat tangents. The curves for $g_1'(0)$ (figure 3) show a rather anomalous behaviour which can, however, be reconciled with (3.14). Their shape is made more plausible in the next section.

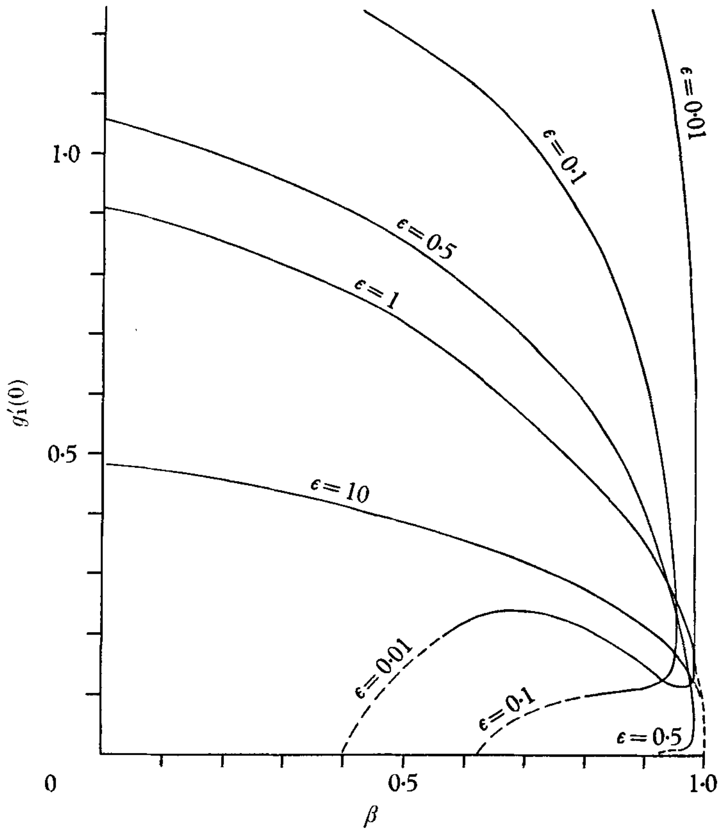

FIGURE 3. Problem I: graphs of $g_1'(0)$ against β for various ϵ.

4. Dual solutions of Problem I. (ii) $\epsilon \to 1$, 0

In this section we examine the properties of β_0 as $\epsilon \to 1$, 0 and from the second we show that duality occurs if $\epsilon \ll 1$.

(*a*) $\epsilon = 1$

Here there is a simple solution of (1.5) subject to (3.9), viz.

$$F_0 = e^{\xi}, \quad G_0 = 1+e^{\xi} \tag{4.1}$$

and clearly

$$\beta_0 = 1. \tag{4.2}$$

(b) $\epsilon \ll 1$

The expansion (3.10) suggests that we write

$$F_0 = \epsilon^{\frac{1}{2}} F_1(\zeta), \quad G_0 = 1 + \epsilon G_1(\zeta), \quad \zeta = \xi\sqrt{\epsilon} \tag{4.3}$$

and consider values of $\zeta = O(1)$. The equations satisfied by F_1, G_1 in the limit $\epsilon \to 0$ are

$$F_1''' + F_1 F_1'' - G_1'' = 0, \quad G_1'' = F_1' \tag{4.4}$$

and, when ζ is large and negative, an appropriate form for the solution is

$$\left.\begin{aligned} F_1 &= e^{\zeta} - \tfrac{1}{6}e^{2\zeta} + \tfrac{5}{144}e^{3\zeta} \ldots, \\ G_1 &= e^{\zeta} - \tfrac{1}{12}e^{2\zeta} + \tfrac{5}{432}e^{3\zeta} \ldots. \end{aligned}\right\} \tag{4.5}$$

The differences between (4.5) and (3.10) can be accounted for by a shift of origin of ζ of amount $\frac{1}{2}\log\epsilon$. As $\zeta \to \infty$ it is consistent to neglect F_1''' in (4.4) whereupon

$$F_1 \simeq \zeta\log\zeta + \zeta\log\log\zeta + C_3\zeta + \ldots, \tag{4.6}$$

C_3 being a constant to be found from a numerical integration of (4.4). The other kind of singularity of F_1, given by (2.3c), is excluded because $F_2'' > 0$ when ζ is large and negative and is therefore always positive. Thus on leaving the zone when $\zeta = O(1)$, F_0' is increasing logarithmically. In order to find its actual limiting value we must therefore adjoin an outer region to this one, in which

$$\lambda = \zeta\phi = \xi\phi\sqrt{\epsilon} \tag{4.7}$$

is of order unity and $\phi \ll 1$. Hence $\lambda = 0$ corresponds to $\zeta \to \infty$ and the boundary conditions on F_0, G_0 as $\lambda \to 0$ are

$$F_0 \simeq \frac{\epsilon^{\frac{1}{2}}\lambda}{\phi}\left[\log\frac{1}{\phi} + \log\log\frac{1}{\phi}\right] + \frac{\epsilon^{\frac{1}{2}}\lambda}{\phi}[\log\lambda + C_3] + \ldots, \tag{4.8}$$

$$G_0 \simeq 1 + O\left(\frac{\epsilon\lambda^2}{\phi^2}\log\frac{1}{\phi}\right). \tag{4.9}$$

A consistent expansion scheme may now be set up on writing

$$\phi = \epsilon^{\frac{1}{2}}[\tfrac{1}{2}\log\epsilon^{-1} + \tfrac{1}{2}\log\tfrac{1}{2}\log\epsilon^{-1}]^{\frac{1}{2}} \tag{4.10}$$

$$F_0 = (\phi/\epsilon^{\frac{1}{2}})\,[F_{2,0} + (1/\log\epsilon^{-1})\,F_{2,1} + \ldots], \tag{4.11}$$

$$G_0 = G_{2,0} + (1/\log\epsilon^{-1})\,G_{2,1} + \ldots,$$

where the F_2, G_2 are functions of λ only. On substituting into (1.5) the leading terms satisfy

$$F_{2,0}F_{2,0}''' = 0, \quad G_{2,0}'' + F_{2,0}G_{2,0}' - F_{2,0}'G_{2,0} = 0, \tag{4.12}$$

with boundary conditions

$$F_{2,0}(0) = 0, \quad F_{2,0}'(0) = 1, \quad G_{2,0}(0) = 1, \quad G_{2\,0}'(0) = 0. \tag{4.13}$$

The appropriate solution of (4.12) is

$$F_{2,0} = \lambda, \quad G_{2,0} = \exp(-\tfrac{1}{2}\lambda^2) + \lambda\int_0^{\lambda}\exp(-\tfrac{1}{2}\lambda_1^2)\,d\lambda_1. \tag{4.14}$$

The next terms can formally be calculated quite easily. Thus $F_{2,1}$ satisfies

$$F_{2,0}F_{2,1}''' = G_{2,0}G_{2,0}'' \tag{4.15}$$

and the boundary condition

$$F_{2,1} \simeq \lambda \log \lambda + C_3 \lambda \quad \text{as} \quad \lambda \to 0. \tag{4.16}$$

Hence $$F'_{2,1} = \exp(-\lambda^2)\log\lambda + 2\int_0^\lambda \lambda_1 \log\lambda_1 \exp(-\lambda_1^2)\, d\lambda_1 + \frac{1}{2}\left[\int_0^\lambda \exp(-\tfrac{1}{2}\lambda_1^2)\, d\lambda_1\right]^2 + C_3 + 1, \tag{4.17}$$

so that $$F'_{2,1}(\infty) = \tfrac{1}{4}\pi - \tfrac{1}{2}\gamma + C_3 + 1, \tag{4.18}$$

where $\gamma = 0{\cdot}577\ldots$ is Euler's constant.

On leaving the range of values of ξ for which $\lambda = O(1)$, it follows from (4.11), (4.14) that

$$F'_0 = dF_0/d\xi \to \epsilon[\tfrac{1}{2}\log\epsilon^{-1} + \tfrac{1}{2}\log(\tfrac{1}{2}\log\epsilon^{-1})]\,[1 + O(\log\epsilon^{-1})^{-1}], \tag{4.19}$$

$$G'_0 = dG_0/d\xi \to \epsilon(\tfrac{1}{2}\pi)^{\frac{1}{2}}\,[\tfrac{1}{2}\log\epsilon^{-1} + \tfrac{1}{2}\log(\tfrac{1}{2}\log\epsilon^{-1})]^{\frac{1}{2}}\,[1 + O(\log\epsilon^{-1})^{-1}] \tag{4.20}$$

and thereafter F'_0, G'_0 remain constant. Hence

$$\beta_0 \simeq \pi/(\log\epsilon^{-1} + \log\tfrac{1}{2}\log\epsilon^{-1}) \quad \text{as} \quad \epsilon \to 0 \tag{4.21}$$

and, as forecast, tends to zero with ϵ. The shape of the graph in figure 1 near $\epsilon = 0$ agrees with (4.21). After the work described in this paper had been carried out, the authors learned that a result similar to (4.21) had already been obtained by Mr M. B. Glauert (private communication, 1963).

This discussion can be used to estimate the behaviour of $g'(0)$ in Problem I as $\beta \to \beta_0$ for $\epsilon \ll 1$. The chief effect of a small change in the initial conditions, when ϵ is small, is to change the value of C_3 in (4.6). Consequently $\Delta C_3 \sim \eta_0^{-2}$ and it then follows from (4.11), (4.18) that the corresponding proportionate changes in $F'_0(\infty)$, $G'_0(\infty) \sim [\eta_0^2 \log\epsilon]^{-1}$ so that the actual change in β, from β_0, $\sim [\eta_0 \log\epsilon^{-1}]^{-2}$. It is concluded that

$$\eta_0^{-1} \sim (\log\epsilon^{-1})\,(\beta - \beta_0)^{\frac{1}{2}}, \tag{4.22}$$

whence in Problem I

$$g'_1(0) \sim \frac{\beta - \beta_0}{\epsilon}[\log\epsilon^{-1}]^{\frac{3}{2}}, \quad f''_1(0) \sim \epsilon^{-\frac{1}{2}}[\log\epsilon^{-1}]^{-\frac{3}{4}} \exp\left\{-\frac{a\sqrt{\epsilon}}{(\beta - \beta_0)(\log\epsilon^{-1})^2}\right\}, \tag{4.23}$$

where a is a positive constant, as $\beta \to \beta_0$ with $\epsilon \ll 1$. It is noted that $g'_1(0)$ has an almost vertical tangent at $\beta = \beta_0$ for small ϵ, which makes more plausible the behaviour of $g'_1(0)$ on the lower portion of the graph displayed in figure 3.

This discussion can also be used to show that duality of solution occurs in Problem I if $\epsilon \ll 1$. For it has already been shown (Glauert 1961) that provided ϵ is sufficiently small there is a solution of Problem I for any $\beta < 1$ in which $g'_1(0) \approx 1$ and we now have found solutions for $\epsilon \ll 1$ in which $g'_1(0) \approx 0$. The numerical results show that the graph of $g'_1(0)$ against β is complicated when $\epsilon \ll 1$ and it is not clear what its shape is in the limit $\epsilon \to 0$.

5. Solutions of Problem II when $\epsilon < 1$

These have been computed numerically for $\epsilon = 0{\cdot}01$, $0{\cdot}1$, $0{\cdot}5$, $1{\cdot}0$ and the corresponding values of $f_2''(0)$, $g_2(0)$ are shown in figures 2 and 4. For a given $\epsilon < 1$, the maximum value of β occurs at β_0, when $f_2''(0) = 0$. This is not surprising, for the problem posed, when $f_2''(0) = 0+$, is identical with that in (3.8) except for the

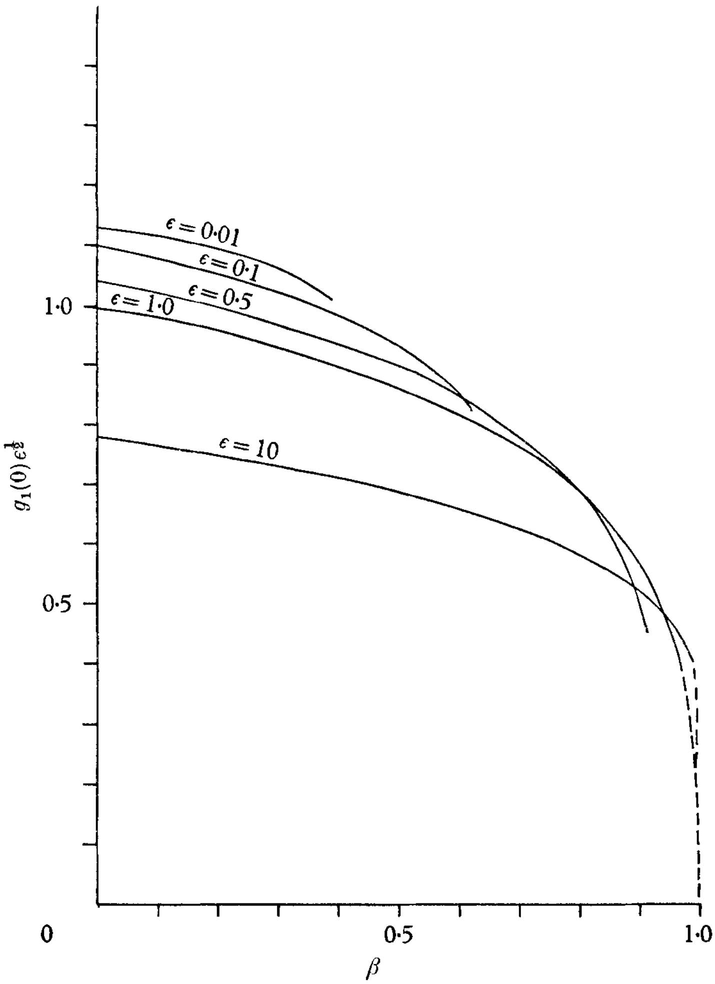

FIGURE 4. Problem II: graphs of $g_2(0)\epsilon^{\frac{1}{2}}$ against β for various ϵ.

(infinite!) shift of origin. The value of $g_2(0)$ at $\beta = \beta_0$ is not zero. Using the notation of (3.8) it is given by

$$g_2(0) = [2/F_0'(\infty)]^{\frac{1}{2}} F_0'(\infty)/G_0'(\infty), \tag{5.1}$$

and we have

$$g_2(0) \simeq 2(\pi\epsilon)^{-\frac{1}{2}}[1+O(1/\log\epsilon^{-1})] \quad \text{as} \quad \beta \to \beta_0 \quad \text{with} \quad \epsilon \ll 1, \tag{5.2}$$

using (4.19), (4.20),

$$g_2(0) \to 0 \quad \text{as} \quad \beta \to \beta_0 \quad \text{with} \quad \epsilon = 1 \tag{5.3}$$

using (4.1). The solutions seem to be unique if $\epsilon < 1$, and as $\beta \to 0$

$$g_2(0) \simeq 2/(\pi\epsilon)^{\frac{1}{2}} \quad \text{if} \quad \epsilon \ll 1, \tag{5.4a}$$

$$g_2(0) \to \simeq 1 \quad \text{if} \quad \epsilon = 1. \tag{5.4b}$$

The reason for (5.4*b*) is not known.

6. Solutions of Problem I when $\epsilon > 1$

Typical solutions have been computed by Carrier & Greenspan (1959) and our object here is to show that, as $\beta \to 1-$, $f_1''(0), g_1'(0) \to 0$. From § 2 we find that if $\epsilon > 1$ there are three possibilities in general: either $f_1 \sim A_1\eta$ as $\eta \to \infty$, or $f_1 \sim A_2\eta^{\alpha_3}$ as $\eta \to \infty$ or $f_1 \sim A_3(\eta^* - \eta)^{\alpha_4}$ as $\eta \to \eta^*-$ where $\alpha_3 > 0 > \alpha_4$ are given by (2.10). Fixing $g_1'(0) = 1$ and letting p vary (see (1.6)), we can expect there to be a critical value p_0, of p, such that if $p > p_0$ the first possibility occurs, the second if $p = p_0$ and the third if $p < p_0$. The number $p_0 \geqslant 0$ is a function of ϵ and from (4.1) we see that $p_0(1) = 0$. If $\epsilon \gg 1$ we can adapt Glauert's (1961) argument to find p_0, as follows. Divide the range of η into two of which the inner has thickness $\sim (\epsilon p)^{-\frac{1}{3}}$. In the inner range write

$$f = \epsilon^{-\frac{2}{3}} p^{\frac{1}{3}} F_3(\chi), \quad g = (\epsilon p)^{-\frac{1}{3}} G_3(\chi), \quad \chi = \eta(\epsilon p)^{\frac{1}{3}}. \tag{6.1}$$

Then (1.5) reduces to

$$F_3'''' = 0, \quad \text{i.e.} \quad F_3 = \tfrac{1}{2}\chi^2 \tag{6.2}$$

and

$$G_3''' + \tfrac{1}{2}\chi^2 G_3' - \chi G_3 = 0, \tag{6.3}$$

with $G_3(0) = 0$, $G_3'(0) = 1$. When χ is large

$$G_3 \simeq 1{\cdot}089(\tfrac{1}{2}\chi^2). \tag{6.4}$$

Subsequently

$$g = 1{\cdot}089\epsilon^{\frac{1}{3}} p^{-\frac{2}{3}} f \tag{6.5}$$

and (1.5) reduces to

$$f'''' + ff''[1 - 1{\cdot}185\epsilon^{\frac{2}{3}} p^{-\frac{4}{3}}] = 0. \tag{6.6}$$

A solution can therefore be found in which $f \sim A_1\eta$ as $\eta \to \infty$ only if

$$p > p_0 = 1{\cdot}136\epsilon^{\frac{1}{2}}. \tag{6.7}$$

The critical solution with $p = p_0$ and $f \sim A_2\eta^{\alpha_3}$ when η is large can be used to find the behaviour of $f_1''(0), g_1'(0)$ as $\beta \to 1-$. When $1 - \beta \ll 1$ the range $0 < \eta < \infty$ is divided into two zones.

(*a*) *Outer zone*, $\eta \sim (1-\beta)^{-\frac{1}{2}}$

Here write

$$f_1 = (1-\beta)^{-\frac{1}{2}} F_4(x), \quad g_1 = f_1 + (1-\beta)^{\frac{1}{2}} H_4(x), \quad \eta(1-\beta)^{\frac{1}{2}} = x. \tag{6.8}$$

Then on substituting into (1.1) we obtain

$$F_4'''' + F_4 F_4''' = F_4 H_4'' + H_4 F_4'', \tag{6.9a}$$

$$F_4'' + \epsilon(F_4 H_4' - H_4 F_4') = 0, \tag{6.9b}$$

neglecting terms proportional to $(1-\beta)^2$. The boundary conditions at infinity are

$$F_4' \to 2, \quad H_4 \to 0, \tag{6.10}$$

and (6.9*b*) can be re-written as

$$H_4 = \frac{F_4}{\epsilon}\int_x^\infty \frac{F_4''}{F_4^2}dx. \tag{6.11}$$

In the solution required, all derivatives of F_4, H_4 are finite and it can therefore only come to an end when $F_4 = 0$. Without loss of generality we can suppose that $F_4 = 0$ at $x = 0$ and that near $x = 0$

$$F_4 = A_4 x^{\alpha_2}, \tag{6.12}$$

where $A_4 > 0$, $\alpha_2 \geqslant 1$ are constants. Then, on substituting into (6.11)

$$H_4 \simeq \alpha_2(\alpha_2 - 1)/\epsilon(\alpha_2 + 1)\,x, \tag{6.13}$$

and on substituting (6.12), (6.13) into (6.9*a*)

$$\alpha_2^2 - \alpha_2 + 2(1+\epsilon)/(1-\epsilon) = 0 \quad \text{or} \quad \alpha_2 = 1; \tag{6.14}$$

(6.13), (6.14) are identical with (2.8), (2.9). The first possibility leads to an acceptable solution if $\epsilon > 1$. The alternative is that $\alpha_2 = 1$, in which case

$$H_4(0) = C_4 \neq 0.$$

For x small it follows that

$$F_4 = A_4 x + \tfrac{1}{2}\epsilon A_4 C_4 x^2 + \ldots, \quad H_4 = C_4 - \frac{\epsilon(\epsilon-1)}{1+\epsilon} C_4^2 x \log\frac{1}{x} + \ldots, \tag{6.15}$$

so that again the condition under which (6.9) was derived fails, this time because $H_4' \to \infty$. In a numerical computation starting from $x = \infty$ there is essentially only one disposable constant, say B_4. Consequently there is a range of values of B_4 such that $F_4 = A_4 x$ near $x = 0$, another in which $F_4 \to \infty$ for finite x as described in (2.10) and a critical value such that (6.12) holds with $\alpha_2 = \alpha_3$ and $F_4' \to 0$ as $x \to 0$. This last solution can be matched to an inner solution of the kind described at the beginning of this section.

(*b*) *Inner zone*

In this zone write

$$f_1 = \theta F_5(y), \quad g_1 = \theta G_5(y), \quad y = \eta\theta, \tag{6.16}$$

where $F_5''(0) = p_0$, $G_5'(0) = 1$ and θ is a scaling factor to be found but which is an order of magnitude larger than $(1-\beta)^{\frac{1}{2}}$, for consistency. Since in this solution $F_5 \neq G_5$ at finite values of y we may replace β in (1.1) by unity and as $y \to \infty$

$$F_5 \simeq A_5 y^{\alpha_3}, \quad F_5 - G_5 \simeq \alpha_3(\alpha_3 - 1)/\epsilon(1+\alpha_3)\,y, \tag{6.17}$$

and we can therefore match f_1, g_1 at $y = \infty$ (6.17) with f_1, g_1 at $x = 0$ (6.12, 6.13) provided only that the scaling factor θ satisfies

$$A_4(1-\beta)^{\frac{1}{2}\alpha_3 - \frac{1}{2}} = A_5\theta^{\alpha_3+1}. \tag{6.18}$$

Hence as $\beta \to 1-$, $\epsilon > 1$,

$$f_1''(0) \simeq \theta^3 F_5''(0) = p_0(A_4/A_5)^{3/(\alpha_3+1)}(1-\beta)^{3(\alpha_3-1)/2(\alpha_3+1)}, \tag{6.19a}$$

$$g_1'(0) \simeq \theta^2 G_5'(0) = (A_4/A_5)^{2/(\alpha_3+1)}(1-\beta)^{(\alpha_3-1)/(\alpha_3+1)}. \tag{6.19b}$$

These results are in agreement with Glauert's (1961) in the limit $\epsilon \to \infty$. As $\epsilon \to 1$, $\alpha_3 \to \infty$ so that the detailed argument is unsatisfactory. However, we note that, using a different approach, Carrier & Greenspan (1959) showed that if $\epsilon = 1$

$$f_1''(0) \sim (1-\beta)^{\frac{3}{2}} [\log (1-\beta)^{-1}]^{\frac{3}{4}} \tag{6.20}$$

as $\beta \to 1-$.

7. Discussion

Consider first the flow of a highly conducting, almost inviscid fluid past a fixed semi-infinite flat plate of zero thickness and in the presence of an aligned field. Here Problem I is relevant and the present state of the theory is that unique solutions can be found for all $\beta < 1$ provided $\epsilon \geqslant 1$. This means that if the Alfvén speed is less than the undisturbed fluid speed, so that outside the boundary layer disturbances cannot travel upstream, then the flow properties can be found. Further, as $\beta \to 1-$, the skin friction and the magnetic field in the plate tend to zero.

If $\epsilon < 1$ then there are two relevant numbers β_0, β_1 such that $0 < \beta_0 < \beta_1 < 1$. For $\beta < \beta_0$ there is also a unique solution while if $\beta > \beta_1$ no solutions can be found. The physical explanation of the non-existence if $\beta_1 < \beta < 1$ is not clear because there is still no upstream propagation of small disturbances possible. It is noted that $\beta_1 > \approx 0{\cdot}95$ for all $\epsilon < 1$ so that this zone is quite narrow. Finally, if $\beta_0 < \beta < \beta_1$, two solutions can be found of which one has an extremely small skin friction and the boundary layer is virtually detached from the wall. Presumably the solution with the larger skin friction and which is the continuation of the solution with $\beta < \beta_0$ occurs in practice.

Secondly, consider the flow of a highly conducting almost inviscid fluid past a non-conducting body in the shape of a thin cylinder and in the presence of a magnetic field parallel to the stream at infinity. Further, it is assumed that outside the boundary layer the velocity and magnetic fields are parallel everywhere and that the magnetic field in the body is zero. This description is controversial (e.g. see Sears & Resler 1959; Stewartson 1963) but since the controversy is still not completely resolved it is of interest to examine the boundary layer associated with it. Assuming that the body is thin but still much thicker than the boundary-layer thickness, Problem II is appropriate. For values of $\epsilon \geqslant 1$ one expects, by analogy with Problem I, that unique solutions can be found for all $\beta < 1$: incidentally the basic theorem of this theory (Hasimoto 1959) was derived for $\epsilon = \infty$ and is of the same kind as the result here. For $\epsilon < 1$ solutions can be found only if $\beta < \beta_0(\epsilon)$ where $\beta_0(1) = 1$, $\beta_0(0) = 0$ so that the model is of limited relevance. In particular, in terrestrial problems $\epsilon \ll 1$ (e.g. $\epsilon \sim 10^{-7}$ for mercury) and so the work of this paper suggests a contradiction for such flows unless the magnetic field is relatively small.

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